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ROCKET BOOSTER CONTROL

SECTION 1

SUMMARY

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ROCKET BOOSTER CONTROL,

SECTION 1 - SUMMARY

~~SUMMARY~~

(NASA Contract NASw-563)

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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ROCKET BOOSTER CONTROL SUMMARY

ABSTRACT

The automatic control research performed by Honeywell for NASA under contract NASw-563 is put into perspective by supplying motivation and indicating the significance of the results obtained. The efforts yielded:

- an algorithm suitable for on-line computation of bounded control amplitude time-optimal controls; the target can be a point set described by a positive definite quadratic form,
- a good method for designing bounded control amplitude and amplitude rate time optimal controls for low order systems,
- procedures for developing bounded phase coordinate, time-optimal controllers,
- a method for measurement of the state for flexible vehicles that does not require mode shape information,
- a method for accurately approximating linear state controllers for flexible vehicles that requires sensors lesser in number than the order of the plant representation,
- a general design method for linear controllers that minimize the effects of load disturbances,
- selectively invariant load disturbance controller design procedures, and
- a method for determination of open loop controllers that minimize the effects of known load disturbances.

Some theoretical results were also obtained; namely:

- a demonstration that formal procedures for eliminating derivatives of control variables yields correct results,
- the development of canonical modes of finite response time: control was carried out using properties of complete controllability,
- a general sufficiency condition for optimal control,
- many results yielding facets for understanding the bounded phase coordinate control problem, and
- an analytical method for assessing the effects of controller design based on a low order plant model.

SUMMARIES

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This discussion separates the efforts into three categories: initial condition control, measurement (of the state vector), and load disturbance control. The efforts in each of the categories are related and then a summary of each effort is provided.

Two appendices are included. Appendix A provides a review of linear programming. Initial considerations leading toward solutions of the load disturbance control problem are presented in Appendix B.

AUTHOR

INITIAL CONDITION CONTROL

Initial condition control is concerned with determining how to operate the control variables so that initial system errors are driven toward zero according to a prescribed criterion. Three theoretical questions were investigated, an algorithm to permit on-line computation of optimal controls was developed, and a number of results for the bounded phase coordinate control problem were obtained.

Mathematical representations for plants quite often naturally contain terms involving derivatives of control variables. With these representations, step changes in the amplitudes of the control variables would require consideration of impulse functions (and perhaps their derivatives). To avoid these difficulties it is common practice (reference 1) to formally perform a transformation that results in a plant representation in which derivatives of the control variable do not appear. Two questions naturally arise: Do the formal transformations yield generally valid results? Could control synthesis be better accomplished within the original representation? Section 2 shows the answer to the first question is yes and indicates the answer to the second is probably not.

The presentation of the modes of finite response time control problem (reference 2) in Section 3 uses the condition of complete controllability to replace a number of special requirements in the original derivation.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control.

An algorithm for determining the controls that minimize the response time between a prescribed point and a prescribed convex target set is presented in Section 5.

Sections 6, 7, 8, 9 and 10 present different attacks on the bounded phase coordinate control problem. An algorithm capable of computing time-optimal, bounded-phase coordinate, two-point boundary value problem trajectories is presented in Section 6; the computer requirements for realizing the controls are demanding. A similar algorithm is presented in Section 8; this algorithm is simpler but can only compute optimal trajectories by the backing out of the origin process. The special bounded phase coordinate problem wherein the constraints are on the control amplitudes and their derivatives is considered in Section 9. A general procedure for backing out of the origin is developed which permits explicit controller design for low order systems. Section 7 considers the existence of solutions to the bounded phase coordinate control problem by use of penalty functions. A reformulation of a bounded phase coordinate problem into an ordinary problem in the calculus of variations is presented in Section 10.

Summaries for Sections 2 through 10 are now presented.

<u>Control Of Plants Whose Representation Contains Derivatives Of The Control Variable</u> Section 2 contains a discussion of problems of control for which the right sides of the plant equations contain derivatives of the control functions. It is a problem of

interest to consider such systems under circumstances in which the controls have discontinuities of the first kind. This requires treating control system equations in which impulse control inputs represented by the derivatives arise. Up to the present time, such systems have been treated by purely formal procedures such as Laning's and Battin's method (reference 1) for eliminating the control derivatives by performing a suitable coordinate transformation. The question naturally arises as to whether such formal procedures yield valid results. Further, if a direct method for treating the control systems while allowing the derivatives of the controls to remain were available, how would the two methods - direct and formal, compare? The contents of Section 2 include the results of a theoretical investigation of the aforementioned direct approach and an example showing how these results can be compared with the results obtained by treating the same system formally.

<u>Modes Of Finite Response Time Control</u> Section 3 considers a linear autonomous system with a single control variable. There are, in general, several modes of finite response time control for such a system. The concepts of single component regulation and multiple component regulation are defined. For completely controllable systems it is shown that multiple component regulation problems can be transformed into single component regulation problems. Thus it is possible to express any of the modes of control considered as control of a single input, single output system.

<u>A Sufficient Condition In Optimal Control</u> The maximum principle

of L. S. Pontriagin and his students is a necessary condition for optimum controls. In many cases this condition is also sufficient. It is shown in Section 4 that in those cases where the system model is linear in the state variables and the cost functional is convex in the state variables this condition is sufficient. The control function may enter in an arbitrary way, but it is assumed that it is bounded in some manner. Use of the condition may require complex calculations.

<u>Time Optimal Control Of Linear Recurrence Systems</u> An algorithm is developed in Section 5 to compute the time-optimal control functions for plants represented by linear recurrence equations. Steering may be to convex sets defined by quadratic forms.

<u>Time-Optimal Bounded Phase Coordinate Control Of Linear Recurrence Systems</u> Section 6 considers the time-optimal control problem with bounded phase coordinates for systems modeled by linear recurrence equations. It is shown that linear inequality constraints of the phase coordinates can be transformed into similar linear inequality constraints on the control variables. The methods for finding the so constrained minimum are discussed. A convergent computational scheme is presented, which, unfortunately, involves a large amount of equipment for implementation. It does not seem practical to solve this problem on line using present computer technology.

<u>Penalty Functions And Bounded Phase Coordinate Control</u> Section 7 considers the use of two different kinds of penalty functions to

obtain approximate and exact solutions to the bounded phase coordinate optimal control problem.

Existence of approximate solutions is established for the cases where the penalty functions are added to the integrand of the performance cost functional. The penalty functions are non-negative, have small values for points well within the phase constraints, but increase rapidly for points near the boundaries of the constraint set.

Optimal solutions obeying the phase constraints are shown, under fairly general hypotheses, to be obtainable by using sequences of penalty functions that converge uniformly to zero on compact sets within the phase constraints and converge uniformly to infinity in compact sets exterior to the phase constraints.

Linear Programming And Bounded Phase Coordinate Control A maximum principle for time-optimal bounded phase coordinate control of ordinary linear differential equation systems and an existence theorem are proven in Section 8. Plant differential equations are then approximated by recurrence equations so the problem of obtaining "optimal" trajectories by backing out of the origin can be reduced to one of linear programming.

The maximum principle permits determination of explicit closed loop control laws for low order systems.

The numerical procedures permit computation of a dense family of "optimal" trajectories through the region of phase space that is of interest for a particular problem. These trajectory data could be stored to provide for closed loop control

Time Optimal Control With Amplitude And Rate Limited Controls Necessary conditions leading to a method for determination of bounded control amplitude and amplitude rate time optimal control trajectories by backing out of the origin are developed in Section 9. The backing out procedure requires choosing the response time, the unaugmented system adjoint vector at the response time, the rate limited control variable amplitudes at the response time, and the rate limited control variable amplitudes at the initial time. A set of consistency conditions on the control variables are then used to determine the allowable control variable trajectories from a finite set of possibilities. The state trajectories including the state at the initial condition can be determined in the usual manner from the control variable trajectories.

The procedures permit computation of a dense family of optimal trajectories through the region of phase space that is of interest for a particular problem. These trajectory data could be stored to provide closed loop control.

A Concise Formulation Of A Bounded Phase Coordinate Control Problem As A Problem In The Calculus Of Variations A short calculation is presented in Section 10 which transforms a bounded phase coordinate control problem into a problem in the calculus of variations. The transformation permits application of the theory and computational methods of variational calculus to the problem of bounded phase coordinate control.

MEASUREMENT

Feedback control implies that variables appearing in the control law be measured. Sections 11, 12, and 13 consider three particular questions regarding measurement problems that arise in applying state control theories to the design of controllers for flexible vehicles. Section 11 provides a method for assessing the errors incurred by controller designs that neglect minor dynamics. If a finite number of flexure modes adequately represent a flexible vehicle, the state can be measured without knowing mode shapes by the method presented in Section 12. Section 13 indicates the number of flexure modes required to model a flexible launch booster is small and applies an existing control synthesis technique to determine the number.

Summaries of Sections 11, 12, and 13 are now presented.

<u>A Note On System Truncation</u> The problem of control design of high order systems based on the characteristics of the lower order essential elements of the system is considered in Section 11 by approximation of asymptotic, sometimes called singular perturbation, type. A preliminary general development is given and detail calculations for a simple problem are written out. These results indicate that the technique leads to the same design found by previous investigators and the only advantage appears to be an estimate of errors.

<u>State Determination For A Flexible Vehicle Without A Mode Shape Requirement</u> If the number of flexure modes required to represent a flexible vehicle is finite, the vehicle dynamics may be

represented by a finite number of first order, ordinary differential equations. It is assumed that n first order equations satisfactorily represent the vehicle dynamics; it is further assumed the coefficients in the equations are known. It is shown that if n different sensors are placed at different (but unknown) positions on the vehicle, the state vector can be determined by processing the outputs of the sensors with a least squares filter.

This method would thus provide a reasonable means for measuring the state of a flexible vehicle if the number of flexure modes were small. The next section (13) indicates the number of flexure modes is small.

An Application Of The Quadratic Penalty Function Criterion To The Determination Of A Linear Control For A Flexible Vehicle In Section 13 linear controllers are designed for a typical five-engine flexible rocket booster by use of the quadratic penalty function criterion. The results show the optimal control can be satisfactorily approximated using 6 sensors.

LOAD DISTURBANCE CONTROL

Load disturbance control is concerned with determining how the control variables should be driven to minimize the effects of load disturbances. Sections 14, 15, and 16 develop synthesis methods to minimize motions due to unknown but bounded disturbances, present a method for preventing motion of certain components of the system state, and develop a method for minimizing motions due to known disturbances.

Summaries of Sections 14, 15, and 16 are now presented.

<u>Minimum Disturbance Effects Control Of Linear Systems With Linear Controllers</u> An optimal control problem for linear systems is considered in Section 14. The optimal controller is one from a class of allowable controllers that minimizes the effect of the worst possible disturbance from the class of uniformly bounded disturbances. When the class of allowable controllers is a family of fixed-gain controllers a method of solution is presented. For sufficiently simple systems the optimal controller may be found by analytical techniques. However, for most realistic systems the use of a computer is required to determine the optimal controller.

As examples, two second order systems are discussed with the optimal controllers being obtained analytically, and a fourth order system is discussed which corresponds to the rigid body representation of a launch booster. This example required the use of a computer to obtain an approximation to the optimal controller. Some discussion of the computation time and computer results is given.

The examples illustrate that a method for the straight forward design of load disturbance controllers has been developed.

<u>An Alternate Derivation And Interpretation Of The Drift-Minimum Principle</u> The design of a control system for a plant with disturbance inputs is considered in Section 15. The motion of the plant is assumed to be described by n linear, first-order constant-coefficient differential equations forced with a scalar control variable and a scalar disturbance. A linear, fixed-gain controller is assumed. In some cases it is possible to choose the gains in

such a manner that a certain type of invariance to disturbances is obtained for the resulting controlled system. Conditions for obtaining such invariance are derived using the concept of complete controllability. The Drift-Minimum condition is obtained as a specific example.

<u>A Minimax Control For A Plant Subjected To A Known Load Disturbance</u> An open-loop optimal control problem is considered in Section 16 for plants that can be represented by linear recurrence equations. It is assumed that the control is bounded and that a known disturbance is present. Then the problem is to choose a control sequence that minimizes an error criterion based on a generalized distance function.

The problem is formulated in a manner such that linear programming techniques can be used to give the optimal control sequence. Estimates on the size of the resulting linear programming problem are presented.

A method is cited for determining an optimal control sequence as a result of varying a nominal disturbance provided the optimal control is known for the nominal case.

An example is presented to illustrate the techniques involved.

The method provides a means for designing open loop controllers that are important in their own right. In addition it provides for determining the effectiveness of closed loop load disturbance controls (such as might be designed by the method of Section 14).

REFERENCES

1. Harvey, C. A., "Measurement Of The State Vector", NASA TN D-1590, 1962

2. Harvey, C. A., "Modes of Control", NASA TN D-1589, 1962

APPENDIX A

A REVIEW OF LINEAR PROGRAMMING TECHNIQUES*

By H. E. Gollwitzer‡

ABSTRACT

The fundamental concepts of linear programming are introduced and the simplex method of computation is briefly discussed. An application of linear programming called parametric linear programming is introduced and briefly developed.

INTRODUCTION

Linear programming is concerned with a special class of maximum and minimum problems. A linear programming problem, in general, is that of maximizing or minimizing a given linear combination of variables subject to a set of linear constraints on these variables. The theory of linear programming gives general results for the existence of optimizing solutions and involved the study of linear inequalities.

The most common method for computing solutions to a linear programming problem is called the simplex method. The simplex method is a computational algorithm which leads to a solution of a linear programming problem in a finite number of iterations. The hypotheses under which the simplex method works are fairly general. The method is used almost exclusively in solving linear programming problems.

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In the following paragraphs the linear programming problem is briefly developed, the simplex method outlined, and a concept called parametric linear programming introduced.

LINEAR PROGRAMMING

The linear programming problem is defined and the necessary notation developed.

DEFINITIONS

The standard linear programming problem is that of finding nonnegative numbers $\xi_i, i=1,\ldots,m$ which either maximize or minimize the given linear combination

$$\sum_{i=1}^{m} \xi_i \gamma_i \qquad (1)$$

subject to the set of linear constants

$$\sum_{i=1}^{m} \xi_i \alpha_{ij} \leq \beta_j, \quad j = 1,\ldots,n. \qquad (2)$$

The linear combination to be maximized or minimized is called the objective function. The term "linear program" is used to denote a linear programming problem when no reference is necessary as to the exact type of problem under consideration. Nonnegative numbers ξ_i, $i = 1,\ldots,m$ which satisfy the constraint conditions are called a feasible solution of the linear program. A feasible solution which maximizes or minimizes the objective function is called an optimal solution. The result of evaluating the objective function with an optimal solution is called the value of the linear program.

A fundamental fact about linear programming is that to every linear program there corresponds another linear program called the dual program. These two programs are closely related and many of the central results in the theory of linear programming are concerned with the relationships between a linear program and its dual. The following example is given to illustrate the manner in which a linear program is related to its dual. The standard maximum program is that of finding nonnegative numbers ξ_i, $i = 1,\ldots,m$ such that

$$\sum_{i=1}^{m} \xi_i \gamma_i \qquad \text{is a maximum}$$

subject to (3)

$$\sum_{i=1}^{m} \xi_i \alpha_{ij} \leq \beta_j \qquad , j = 1,\ldots,n.$$

The dual program is to find numbers η_j such that

$$\sum_{j=1}^{n} \eta_j \beta_j \qquad \text{is a minimum}$$

subject to (4)

$$\sum_{i=1}^{n} \eta_j \alpha_{ij} \geq \gamma_i \qquad , i = 1,\ldots,m,$$

where the numbers η_j are unrestricted in sign.

VECTOR AND MATRIX THEORY

The form of linear programming problems leads to the shorthand of vector and matrix notation. If A is an mxn matrix, then A' denotes the transpose of A. Vectors will be considered as row vectors and an n-vector x will denote the n-tuple $(x_1,\ldots,x_n)$. A vector x is nonnegative if and only if each of its components are nonnegative. The inner product of two n-vectors x and y is denoted as xy. In vector and matrix notation the standard maximum problem and its dual are formulated as follows: Given an mxn matrix A, an n-vector b, and an m-vector c, find a nonnegative vector x such that

$$xc \qquad \text{is a maximum}$$

subject to (5)

$$xA = b$$

and the dual problem is that of finding an n-vector y such that

$$yb \qquad \text{is a minimum}$$

subject to (6)

$$Ay \geq c$$

Let $M = \{1,\ldots,m\}$ denote the set of positive integers from 1 to m. Let $a_1,\ldots,a_m$ be m-vectors and let S be a subset of M. A solution $x = \{\xi_1,\ldots,\xi_m\}$ of the equation

$$\sum_{i=1}^{m} \xi_i a_i = b \tag{7}$$

is said to depend on the set S if $\xi_i = 0$ for $i \notin S$. A solution of (7) is called a basic solution if it depends on a set S such that the m-vectors a_i are linearly independent for $i \in S$. A basic nonnegative solution is defined similarily.

THE CANONICAL PROBLEM

One type of linear program which is quite useful in connection with the simplex method is that in which the unknown vector is non-negative and the constraints are equations. The type of problem is called the canonical problem. The canonical minimum problem is given as follows:

Given an mxn matrix A, an n-vector b, and an m-vector c, find a nonnegative vector x such that

$$xc \qquad \text{is a minimum}$$

subject to (8)

$$xA = b.$$

The canonical maximum problem is similarily defined.

The canonical program and the standard program are equivalent in the sense that either problem can be transformed in the other. For example, the canonical maximum program is that of finding a nonnegative vector x such that

$$xc \qquad \text{is a maximum}$$

subject to

$$xA = b.$$

The equivalent standard program is that of finding a nonnegative vector x such that

$$xc \qquad \text{is a maximum}$$

subject to

$$xA \leq b$$

$$-xA \leq -b.$$

The standard maximum program is that of finding a nonnegative vector x such that

$$xc \qquad \text{is a maximum}$$

subject to

$$xA \leq b.$$

This is transformed into a canonical program by adjoining one new nonnegative variable to each constraint so that equality is always preserved. In matrix notation the problem is to find a nonnegative vector x such that

$$xc \qquad \text{is a maximum}$$

subject to

$$xA + zI = b$$

where I is the mxm identity matrix and z is a nonnegative m-vector. The components of z are sometimes called slack variables. The components of z do not enter into the objective function and serve only to preserve equality in each of the constraints. Finally, it is noted that any maximum problem can be treated as a minimum problem by negating the objective function, and conversely.

The following example will indicate how a fairly general linear program can be put into canonical minimum form. This will be done in three main steps. Find numbers ξ_1, ξ_2, ξ_3 that maximize $8\xi_1 + 19\xi_2 + 7\xi_3$ subject to

$$\begin{aligned} 3\xi_1 + 4\xi_2 + \xi_3 &\leq 25 \\ \xi_1 + 3\xi_2 + 3\xi_3 &\leq 50 \\ \xi_1 - \xi_2 + \xi_3 &\geq 0 \end{aligned} \tag{9}$$

The first step is to get all variables expressed as the difference of two nonnegative variables. The next step is to negate the objective function so that a minimum problem can be considered. Let $\xi_1 = x_1 - x_2$, $\xi_2 = x_3 - x_4$, $\xi_3 = x_5 - x_6$.

With these two steps carried out the problem now reads;

Find nonnegative numbers x_i, $i = 1,\ldots,6$

such that

$$-8x_1 + 8x_2 - 19x_3 + 19x_4 - 7x_5 + 7x_6$$

is a minimum subject to

$$\begin{aligned}
3x_1 - 3x_2 + 4x_3 - 4x_4 + x_5 - x_6 &\leq 25 \\
x_1 - x_2 + 3x_3 - 3x_4 + 3x_5 - 3x_6 &\leq 50 \\
-x_1 + x_2 + x_3 - x_4 - x_5 + x_6 &\leq 0
\end{aligned}$$

The third step is to add the nonnegative slack variables x_7, x_8, x_9 so that the final canonical minimum problem is to find nonnegative variables x_i, $i = 1,\ldots,9$ such that $-8x_1 + 8x_2 - 19x_3 + 19x_4 - 7x_5 + 7x_6$ is a maximum subject to

$$\begin{aligned}
3x_1 - 3x_2 + 4x_3 - 4x_4 + x_5 - x_6 + x_7 \qquad\qquad &= 25 \\
x_1 - x_2 + 3x_3 - 3x_4 + 3x_5 - 3x_6 \qquad + x_8 \qquad &= 50 \\
-x_1 + x_2 + x_3 - x_4 - x_5 + x_6 \qquad\qquad + x_9 &= 0
\end{aligned}$$

If a solution for this problem is obtained, the resulting x_i's will permit determination of the optimal ξ_i's in (9).

THE SIMPLEX METHOD

The simplex method is an iterative procedure for finding an optimal solution to a linear program if one exists. The basic algorithm involves lengthy arithmetic calculations at each iteration so it was not until the advent of large, high-speed digital computers that large linear programs could be solved.

The simplex method is designed around the canonical minimum or maximum problem and thus it may be necessary to transform the given problem into a canonical form. The problem is to find a nonnegative vector x such that

$$xc \quad \text{is a minimum} \tag{10}$$

subject to

$$Ax = b \tag{11}$$

where x is an n-vector, c an n-vector, b an m-vector and A an mxn matrix. If the equation (11) is considered as a vector identity, then the problem is to find a nonnegative vector $x = (x_1,\ldots,x_n)$ such that

$$xc \text{ is a minimum}$$

subject to

$$\sum_{i=1}^{n} x_i a^i = b \tag{12}$$

where $a^i = (a_{1i}, a_{2i},\ldots,a_{mi})$ and a_{ij} is the ij^{th} component of A. In a canonical program an optimal vector $\overline{x}$ is called a basic optimal vector if it is a basic solution of the constraint equation $Ay = b$. From the theory of linear programming, if a canonical linear program has an optimal vector, it has a basic optimal vector. This necessary condition yields a proof that a canonical linear program can be solved in a finite number of steps. The proof follows because there are only a finite number of basic solutions to (12) and hence certainly a finite number of nonnegative basic solutions. It would be time consuming to check every possible basic solution in order to find the nonnegative basic solution that optimizes the linear program. The simplex method gives an orderly method for searching for the basic optimal vector. The main idea is to first find a nonnegative basic solution of (11) and to then find another nonnegative basic solution that reduces the value of the objective function. The purpose of this is to eliminate the testing of nonnegative basic solutions that do not decrease the value of the objective function. Since there are only a finite number of nonnegative basic solutions, this search will eventually find the optimal basic solution if it exists.

There is also the question of whether this process always works. In some cases the simplex method may fail to give an optimal solution. These cases are called degenerate. Degeneracy occurs when the simplex algorithm becomes ambiguous to use. There are methods for handling degenerate cases so that an optimal solution can be found (reference 1).

In the discussions of the simplex method the assumption was made that a nonnegative basic solution was known. This initial nonnegative basic solution can be found in the following manner. The original problem was to find a nonnegarive n-vector x such that

$$xc \quad \text{is a minimum} \tag{10}$$

subject to

$$Ax = b. \tag{11}$$

The vector b is assumed to be nonnegative, and can be accomplished by changing the signs of the necessary equations. Let $\overline{c} = (1,1,\ldots,1)$ be an m vector and let y be a nonnegative m vector. The problem of finding a nonnegative basic solution of (11) is that of finding nonnegative vectors x and y such that

$$y\bar{c} \quad \text{is a minimum} \tag{13}$$

subject to

$$Ax + yI = b \tag{14}$$

where I is the identity matrix. A theorem in linear program states that $Ax = b$ has a nonnegative basic solution if and only if the value of (13) is zero. This gives the initial nonnegative basic vector, and hence it is necessary to solve a linear program to get an initial nonnegative basic vector. It is then necessary to obtain an initial nonnegative basic vector to start the solution of (13), (14). This is accomplished by setting $x = 0$ and $y = b$, which is a nonnegative basic vector since b was nonnegative. If the minimum value of (13) is not zero then the original program does not have a nonnegative basic solution and hence no optimal solution. This affords a quick check on the solvability of a linear program. This concludes the discussion of the simplex method.

PARAMETRIC LINEAR PROGRAMMING

The method of parametric linear programming is a technique that permits the computation of new optimal solutions, as a result of changes in the data, without solving the entire linear program again. This is best illustrated by considering the following standard canonical minimum problem: Find nonnegative numbers x_i, $i = 1,\ldots,n$ that

$$cx \quad \text{is a minimum} \tag{15}$$

subject to

$$Ax = b. \tag{16}$$

It is of interest to investigate the changes in an optimal solution if say, the components of the vector b are changed in a prescribed manner. It would be advantageous to have a method that would give changes in the optimal solution as a function of changes in the components of b without solving another linear program. This is a consequence of the large amount of effort needed to solve linear programs. Parametric linear programming is one such method.

Let

$$\xi' = (\xi_1', \ldots, \xi_n') \tag{17}$$

be an optimal solution to (16). The solution will depend on m linearly independent column vectors, and the remaining ξ_j' will be zero. Assume for convenience that the solution depends on the first m column vectors in (16). This is sufficient because any other arrangement can be relabeled to give this form. Then $\xi' = (\xi_1', \ldots, \xi_m', 0, \ldots, 0)$. The equation (15) will appear as

$$\begin{array}{ccccc} a_{11}\xi_1' + & \cdots & + a_{1m}\xi_m' & = & b_1 \\ \vdots & & \vdots & & \vdots \\ a_{m1}\xi_1' + & \cdots & + a_{mm}\xi_m' & = & b_m \end{array} \tag{18}$$

where the a_{ij} in (18) can be different from the a_{ij} in (16) because of relabeling. Letting $\Gamma = (\gamma_{ij})$ be the matrix of coefficients in (18), and $\xi = (\xi_1', \ldots, \xi_m')$, it can be written as

$$\Gamma\xi = b \; . \tag{19}$$

Since the columns of Γ are linearly independent Γ is non-singular and $\xi = \Gamma^{-1}b$, where Γ^{-1} denotes the inverse of Γ. The optimal solution ξ can then be expressed as

$$\xi_i' = \sum_{j=1}^{m} b_j \beta_{ij}, \qquad i = 1,\ldots,m, \tag{20}$$

where β_{ij} is the ij element of Γ^{-1}.

Equation (20) expresses the optimal solution in terms of the known vector b and the matrix Γ. Since the solution is optimal, $\xi_i' \geq 0$, $i = 1,\ldots,m$, and as long as this condition holds the solution will remain optimal with respect to the chosen basis (reference 2). Let the b_j be allowed to vary by an amount $\rho_j\theta$, where θ is a continuous variable, $\theta \geq 0$ and ρ_j a given constant. Define $b_{j_\theta} = b_j + \rho_j\theta$, $j = 1,\ldots,m$. The solution ξ will remain feasible and hence optimal with respect to the chosen basis if

$$\xi_{i_\theta}' = \sum_{j=1}^{m} b_{j_\theta} \beta_{ij} \geq 0, \qquad i = 1,\ldots,m, \tag{21}$$

where ξ_{i_θ}' is the value of the i^{th} component of the optimal solution when the components of b are varied. Now if $\sum_{j=1}^{m} \rho_j \beta_{ij} \geq 0$, $i = 1,\ldots,m$, the solution is termed "open", as θ can be extended to any positive value and the program will remain feasible and hence optimal. But if $\sum_{j=1}^{m} \rho_j \beta_{ij} < 0$ for some i, then if θ is allowed to exceed a certain value θ_{max}, the solution is no longer feasible and hence not optimal. This maximum value can be determined if the solution is not "open" from the constraint (21). This value of θ_{max} is given by

$$\theta_{max} = \min_{A} \frac{\sum_{j=1}^{m} \beta_{ij} b_j}{\left| \sum_{j=1}^{m} \beta_{ij} \rho_j \right|} \tag{22}$$

where $A = \left\{ i \,\middle|\, \sum_{j=1}^{m} \beta_{ij}\rho_j < 0 \right\}$. The value of θ_{max} is determined by letting equality hold in (21) for $i \in A$. Since there is some i for which this maximum occurs, if θ is increased by ϵ, $\epsilon > 0$, the variable ξ_i will go negative. Thus if it is desired to increase θ beyond θ_{max}, ξ_i must be replaced by some variable not yet in the present basis. This replacement process will not be given but is explained in reference 2.

In conclusion, if an optimal solution is known then certain changes in the statement of the problem can be adequately handled without solving an entirely new problem.

CONCLUSIONS

The concept of linear programming has been introduced, and the philosophy of the simplex method briefly explained. The technique of parametric linear programming has been introduced and briefly developed.

REFERENCES

1. Gale, David, "Theory of Linear Economic Models", McGraw-Hill Book Company, Inc., New York, 1960.

2. Garvin, Walter W., "Introduction To Linear Programming", McGraw-Hill Book Company, Inc., New York, 1960.

APPENDIX B

CONSIDERATIONS IN LOAD

DISTURBANCE CONTROL

By C. A. Harvey

Two initial results are presented which were obtained in attempts to gain some understanding of the following optimal control problem. Suppose that the system is represented by the vector differential equation

$$\dot{x} = Ax + Bu + Dg,\ x(0) = 0, \quad 0 \leq t \leq T < \infty \qquad (1)$$

where x is an n-vector representing the state of the system,

u is an m-vector representing the control variables,

g is an r-vector representing a disturbance, and

A, B, and D are continuous matrices of appropriate size.

The vector g is assumed to be a member of some prescribed class of disturbances G. Also, it is assumed that the control u is a member of a specified allowable class of controls U. The optimization problem is then to find a control u in the class U that minimizes C(u) when C(u) is defined by

$$C(u) = \max_{1 \leq i \leq n} \ \max_{g \in G} \ \max_{0 \leq t \leq T} |x_i(t;u,g)|$$

where $x_i(t;u,g)$ represents the i^{th} component of the solution to (1).

The first result deals with the uniqueness of optimal controls. If the class, G, does not contain disturbances with enough variations then the optimal control is not unique. A very trivial example is that which occurs when G is empty; then any control

$u(x,t)$ with $u(0,t) = 0$ is optimal. A somewhat less trivial example is the following: Let the system represented in the form (1) be given by $A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$, $B = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$, $D = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$. Let T be any positive real number and take $G = \{g(t) : g(t) = c,\ 0 \leq t \leq T,\ |c| \leq 2\}$ and take U to be all measurable functions of x_1 and x_2 that satisfy the inequality, $|u(x_1,x_2)| \leq 1$. In this case $u(x_1,x_2) = \operatorname{sgn} f(x_1,x_2)$ is optimal for any $f(x_1,x_2)$ that satisfies the conditions

$$f(0,0) = 0$$

$$f(x_1,x_2) > 0 \text{ when } x_1 > 0 \text{ and } x_2 > 0$$

$$f(x_1,x_2) < 0 \text{ when } x_1 < 0 \text{ and } x_2 < 0$$

The other result concerns a conjecture that the optimal control may be obtained as an optimal control for some initial condition problem. Two problem statements are introduced to make possible a precise statement of the conjecture.

P_1: Consider the system $\overset{\circ}{x} = Ax + bu$ where x is an n-vector, the system state, u is a scalar, the control and A is an nxn matrix, b an n-vector. Let $x(t;\, x_o, u)$ denote the solution to the system with initial conditions $x(0) = x_o$ and control u. The problem then is to find $u(x)$ satisfying certain restrictions such that

$$\max_{1 \leq i \leq n} \left[\max_{x_o \in \Omega} \left(\max_{0 \leq t \leq T^*} |x_i(t; x_o, u)| \right) \right] \quad \text{is a minimum}$$

where Ω is some subset of the Euclidean n-space and $T^* > 0$ (possibly infinite).

P_2: Consider the system $\overset{\circ}{x} = Ax + bu + g(t)$, where $g(t)$ is an n-vector belonging to a class G; x, u, A, and b are as in P_1. The problem is to find $u(x)$ satisfying the same restrictions as in P_1 such that

$$\max_{g(t)\epsilon G} \left[\max_{i\leq i\leq n} \left(\max_{0\leq t\leq T} | x_1(t;0,u,g)| \right) \right] \text{ is a minimum}$$

where $x(t;0,u,g)$ denotes the solution of $\dot{x} = Ax + bu + g$ with $x(0) = 0$.

The conjecture is that a solution to P_1 is a solution to P_2. This conjecture is really quite vague in the sense that the set Ω in P_1 is not completely defined and that T^* in P_1 is not completely specified. These would have to be related to T and G in the statement of P_2.

The following example is presented as a partial result concerning the conjecture. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $u(x)$ be of the form $x_1(\cos\theta) + x_2(\sin\theta)$ with the parameter θ being in the interval $[0,2\pi)$. Then the problem related to P_1 is:

P_1^*: with Ω chosen to be a closed circular disc and T^* chosen to be infinite, find $\theta \epsilon [0,2\pi)$ such that

$$J_1(\theta) = \max_{x_o\epsilon\Omega} \left(\max\left[\max_{t\geq 0} |x_1(t;x_o,u)|,\ \max_{t\geq 0}|x_2(t;x_o,u)|\right]\right)$$

is a minimum where $\Omega = \left\{(x,y):x^2+y^2 \leq R^2\right\}$ and $u = x_1(\cos\theta)+x_2(\sin\theta)$.

The problem related to P_2 is:

P_2^*: Find $\theta\ \epsilon[0,2\pi)$ such that

$$J_2(\theta) = \max_{g\epsilon G}\left(\max\left[\max_{0\leq t\leq T} |x_1(t;0,u,g)|,\ \max_{0\leq t\leq T} |x_2(t;0,u,g)|\right]\right)$$

is a minimum where T is a sufficiently large positive real number, $u = x_1(\cos\theta) + x_2(\sin\theta)$, and $G = \left\{g(t):g(t)=c, |c|\leq R^2\right\}$.

It is possible to solve P_2^* to find the optimal θ, say θ^o.

The value of θ^o is approximately 209.4^o and $J_2(209.4^o)$ is approximately 1.635 K^2. To show this it may be established that for any θ satisfying $0 \leq \theta < \pi$ or $3\pi/2 \leq \theta < 2\pi$ the solution corresponding to $g(t) = K^2$ on the interval $[0,\infty)$ is unbounded. Hence for T sufficiently large $J_2(\theta) > 2K^2$ if θ satisfies $0 \leq \theta < \pi$ or $3\pi/2 \leq \theta < 2\pi$. Let θ_o be in the interval $(\pi, 3\pi/2)$ and satisfy the equation $\cos \theta_o = 2 - \sqrt{5}$. For any θ satisfying $\theta_o \leq \theta < 3\pi/2$ the first component of the solution corresponding to $g(t) = K^2$ approaches $K^2 (-\cos \theta)^{-1}$ as t approaches infinity. But $K^2 (-\cos \theta)^{-1} > 4K^2$ for $\theta_o \leq \theta < 3\pi/2$ and hence $J_2(\theta) > 2K^2$ if T is sufficiently large. For θ in the interval $[\pi, \theta_o)$ it is possible to obtain as T approaches infinity that

$$J_2(\theta) = K^2 (-\cos \theta)^{-1}\left\{1 + \exp [\alpha(\theta)\pi/\beta(\theta)]\right\}$$

where $\alpha = \frac{1}{2} \sin \theta$ and $\beta(\theta) = \frac{1}{2}\sqrt{-\sin^2\theta - 4\cos\theta}$. This expression assumes an absolute minimum for $\theta = \theta^o$. For T sufficiently large, θ^o will also minimize $J_2(\theta)$.

Now for the problem P_1^*, this value of θ^o is not optimal. In fact, there is no value of θ in the interval (π, θ_o) which is optimal for P_1^*. Thus the problem P_1^* is not equivalent to P_2^*. It is possible that choosing a different Ω for the problem P_1^* may lead to a problem which has the same solution as P_2^*.